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The Faber Operator and its Boundedness

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Let G be a domain bounded by a Jordan curve Γ, and let $A(\bar{G})$ be the Banach space of functions continuous on $\bar{G}$ and holomorphic in G. The Faber operator T is a linear mapping from $A(\bar{\mathbb{D}})$ to $A(\bar{G})$ mapping w^n onto the nth Faber polynomial $F_n(z)$ $(n=0, 1, 2, ...)$. We show that $\|T\| < \infty$ if Γ is piecewise Dini-smooth, and give an example of a quasicircle Γ for which $\|T\| = \infty$.

Key Words: complex approximation; Faber operator.

1. INTRODUCTION AND MAIN RESULTS

In the following G is a domain in $\mathbb{C}$ bounded by a Jordan curve Γ, and $A(\bar{G})$ is the Banach space of functions F which are holomorphic in G and continuous on $\bar{G}$; we let $\|F\| = \max\{|F(z)| : z \in \bar{G}\}$. If G is the unit disk $\mathbb{D}$, we get the Banach space $A(\bar{\mathbb{D}})$. Given $F \in A(\bar{G})$, our problem is to find estimates for

$$E_n(F, \bar{G}) := \min\{\|F-P\| : P \in \Pi_n\}$$

where Π_n is the set of all polynomials of degree $\leq n$. This is a classical problem; see for example Gaier [6] or Smirnov–Lebedev [12] and references given there.

One elegant method to achieve this is the use of the Faber polynomials F_n and the Faber operator T associated with the domain G. Assume we have such an operator T with the following properties:

(i) T maps w^n onto $F_n(z)$ $(n=0, 1, 2, ...)$;

(ii) T is linear and bounded on $\Pi = \bigcup_{n=0}^{\infty} \Pi_n \subset A(\bar{\mathbb{D}})$ and can therefore be extended to a linear and bounded map from $A(\bar{\mathbb{D}})$ to $A(\bar{G})$;

(iii) given $F \in A(\bar{G})$, there is an $f \in A(\bar{\mathbb{D}})$ with $F = Tf$.

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Then we have, for arbitrary coefficients a_k,

$$F-\sum_{k=0}^{n} a_k F_k = T\left(f-\sum_{k=0}^{n} a_k w^k\right)$$

and

$$\left\| F-\sum_{k=0}^{n} a_k F_k \right\| \leqq \|T\| \cdot \left\| f-\sum_{k=0}^{n} a_k w^k \right\|,$$

from which it follows that

$$E_n(F, \bar{G}) \leqq \|T\| \cdot E_n(f, \bar{\mathbb{D}}), \tag{1.1}$$

so that the original problem is reduced to an approximation problem in $\bar{\mathbb{D}}$.

It is therefore important to know which conditions on Γ imply $\|T\| < \infty$. We deal with this question in Sections 3 and 4. We give a new geometric criterion for $\|T\| < \infty$ and an example of a domain with $\|T\| = \infty$.

THEOREM 1. *If Γ is piecewise Dini-smooth, then $\|T\| < \infty$.*

A subarc γ of Γ, $z = z(s)$ (where $s \in [a, b]$ is arc length) is called *Dini-smooth* if γ is smooth, i.e. $z'(s)$ is continuous in $[a, b]$, and if furthermore $z'(s)$ has a modulus of continuity ω which satisfies

$$\int_0^c \frac{\omega(t)}{t}\, dt < \infty \qquad \text{for some} \quad c > 0. \tag{1.2}$$

Equivalently, the tangent angle $\vartheta = \vartheta(s) = \arg z'(s)$ will have a modulus of continuity satisfying (1.2). And Γ is called *piecewise* Dini-smooth if $\Gamma = \bigcup \gamma_j$ with a finite number of Dini-smooth arcs γ_j. Here Γ may have corners and cusps.

THEOREM 2. *There is a domain G with quasiconformal boundary Γ for which $\|T\| = \infty$.*

This will be an analytical construction using the exterior mapping function ψ. We do not know of a purely geometric way to construct such a Jordan curve Γ.

2. THE FABER POLYNOMIALS AND THE FABER OPERATOR

In the following we give some definitions and survey known results.

2.1. *Jordan Curves of Bounded Secant Variation*

If Γ is rectifiable, $z = z(s)$ with arc length $s \in [0, L]$, and if $\vartheta(s) := \arg z'(s)$ can be defined on $[0, L]$ to become a function of bounded variation, then Γ is called of *bounded rotation* ($\Gamma \in BR$), and $\int_\Gamma |d\vartheta(s)|$ is called the total rotation of Γ.

For our purposes a larger class of Jordan curves is important. We consider the function $h(\zeta) := \arg(\zeta - z)$ for fixed $z \in \Gamma$ or $z \in G$, and where ζ traverses Γ. If $z = z(s)$ is on Γ, ζ starts at $z(s+)$ and stops at $z(s-)$; the total variation of $h(\zeta)$ as a function of ζ will be denoted by $\operatorname{Var}_\zeta \arg(\zeta - z)$. If this is finite, it is clear that $\arg(\zeta - z)$ has limits as $\zeta \to z(s+)$ and as $\zeta \to z(s-)$: Γ possesses forward and backward tangents at z.

Definition. If there is a fixed constant M such that

$$\operatorname{Var}_\zeta \arg(\zeta - z) \leqq M < \infty \qquad \text{for all} \quad z \in \Gamma,$$

then Γ is called of bounded secant variation. We write $\Gamma \in BSV$.

This class of Jordan curves was introduced by Andersson [2], see also Korevaar [8]. Andersson showed that $\Gamma \in BR$ implies $\Gamma \in BSV$ but not conversely. Furthermore, it is not difficult to construct a smooth Γ which is not of BSV.

If $z \in G$, the total variation of $h(\zeta)$, as ζ traverses Γ, is independent of the starting point, and will be denoted by

$$\operatorname{Var}_\zeta \arg(\zeta - z), \qquad z \in G.$$

By way of an example, take Γ to be the unit circle. We get

$$\operatorname{Var}_\zeta \arg(\zeta - 1) = \pi \qquad \text{and} \qquad \operatorname{Var}_\zeta \arg(\zeta - 0) = 2\pi.$$

Lemma 1. *If Γ is of bounded secant variation,*

$$\operatorname{Var}_\zeta \arg(\zeta - z) \leqq M < \infty \qquad \textit{for all} \quad z \in \Gamma, \tag{2.1}$$

then

$$\operatorname{Var}_\zeta \arg(\zeta - z) \leqq M + 2\pi \qquad \textit{for all} \quad z \in G. \tag{2.2}$$

Proof. Let $\zeta_0, \zeta_1, ..., \zeta_j, \zeta_{j+1}, ..., \zeta_N = \zeta_0$ be N different points on Γ in positive orientation. We study

$$h_N(z) := \sum_{j=0}^{N-1} |\arg(\zeta_{j+1} - z) - \arg(\zeta_j - z)|$$

for $z \in G$. This is a subharmonic function in G, and since Γ has half-tangents at each point ζ_j, each term $\arg(z - \zeta_j)$ is bounded in G, so that each h_N is subharmonic *and bounded* in G. Now let $z \to z_0 \in \Gamma$, $z_0 \neq \zeta_j$ $(j = 1, 2, ..., N)$. Assume that z_0 is on an arc from ζ_{j_0} to ζ_{j_0+1}. It is clear that

$$h_N(z) \to \sum_{j \neq j_0} |\arg(\zeta_{j+1} - z_0) - \arg(\zeta_j - z_0)| + \alpha \tag{2.3}$$

where α is the angle at z_0 of the triangle ζ_{j_0}, z_0, ζ_{j_0+1} and thus $0 \leqq \alpha \leqq 2\pi$, while the sum in (2.3) is $\leqq M$ by assumption. We get

$$\overline{\lim}\, h_N(z) \leqq M + 2\pi \qquad \text{as} \quad z \to z_0 \in \Gamma, \quad z_0 \neq \zeta_j.$$

Lindelöfs maximum principle for subharmonic functions (Ahlfors [1], p. 38 or Heins [7], p. 76) now gives $h_N(z) \leqq M + 2\pi$ for all $z \in G$, and (2.2) is established. ∎

2.2. *The Faber Polynomials*

We collect a few known facts; see for example [6], p. 46ff. If

$$z = \psi(w) = bw + b_0 + \frac{b_1}{w} + \cdots \qquad |w| > 1 \tag{2.4}$$

is the normalized exterior mapping function which maps $\{w: |w| > 1\}$ onto the exterior of the Jordan curve Γ, the Faber polynomials can be defined by a generating function:

$$\frac{w\psi'(w)}{\psi(w) - z} = 1 + \sum_{n=1}^{\infty} F_n(z)\, w^{-n}, \qquad |w| > 1, \quad z \in \overline{G}. \tag{2.5}$$

From this follows an integral representation

$$F_n(z) = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} w^n \frac{\psi'(w)}{\psi(w) - z}\, dw, \qquad z \in G; \quad n = 0, 1, 2, ... \tag{2.6}$$

provided that Γ is rectifiable so that ψ' is integrable on $\partial\mathbb{D}$. Another integral representation

$$F_n(z) = \frac{1}{\pi} \int_{t=0}^{2\pi} e^{int}\, d_t \arg[\psi(e^{it}) - z], \qquad z \in \Gamma; \quad n = 1, 2, ... \tag{2.7}$$

was proved by Pommerenke [9], p. 425 whenever $\Gamma \in BR$, but (2.7) is actually true for the wider class $\Gamma \in BSV$; see Andersson [2], p. 4.

Finally, we note that there is a direct relation between the coefficient b_n in (2.4) and the Faber polynomial F_n:

$$\frac{nb_n}{b} = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} F_n(\psi(\omega))\, d\omega \qquad n = 0, 1, 2, ... \tag{2.8}$$

In Pommerenke [10], p. 58 this is shown via the Grunsky coefficients, but these can be avoided by integrating (2.5) with $z = \psi(\omega)$ on $\partial\mathbb{D}$ and applying the residue theorem.

2.3. *The Faber Operator*

Motivated by (2.6), we can give an integral representation of the Faber operator T by

$$(Tf)(z) := \frac{1}{2\pi i} \int_{\partial\mathbb{D}} f(w)\, \frac{\psi'(w)}{\psi(w) - z}\, dw \qquad z \in G, \tag{2.9}$$

provided that Γ is rectifiable. The function $F = Tf$, for $f \in A(\bar{\mathbb{D}})$, will be holomorphic in G, but if T is a bounded operator, i.e. if there is a constant C such that

$$\sup\{|F(z)| : z \in G\} \leqq C \cdot \sup\{|f(w)| : w \in \mathbb{D}\} \tag{2.10}$$

holds for all $f \in A(\bar{\mathbb{D}})$, then the image function F will be in the subspace $A(\bar{G})$ of Hol G, and T satisfies the assumptions (i) and (ii) of the introduction.

To obtain (2.10), we bring (2.9) into different form. For this, we need a lemma.

LEMMA 2. *Let g be continuous on $\partial\mathbb{D}$, and $h \in L^1$ on $\partial\mathbb{D}$. Assume that*

$$g(e^{it}) \sim \sum_{k \geqq 0} a_k e^{ikt} \qquad \text{and} \qquad h(e^{it}) \sim \sum_{k \geqq 0} b_k e^{ikt}.$$

Then

$$\frac{1}{2\pi i} \int_0^{2\pi} g(e^{it})\, f(e^{it})\, dt = -ia_0 b_0. \tag{2.11}$$

Proof. Let

$$g(z) := \sum_{k=0}^{\infty} a_k z^k \in A(\bar{\mathbb{D}}) \qquad \text{and} \qquad h(z) := \sum_{k=0}^{\infty} b_k z^k \in H^1(\mathbb{D})$$

be the holomorphic extensions of g and h into $\mathbb{D}$. The residue theorem gives

$$\frac{1}{2\pi i}\int_0^{2\pi} g(e^{it})\,h(e^{it})\,dt = \frac{1}{2\pi i}\int_{|z|=r<1} g(z)\,h(z)\,\frac{dz}{iz}$$
$$= \frac{1}{i}\,g(0)\,h(0) = -ia_0 b_0. \quad \blacksquare$$

Now put $w = e^{it}$ in (2.9) to get

$$F(z) = \frac{1}{2\pi i}\int_0^{2\pi} f(e^{it})\left[\frac{\psi'(e^{it})\,ie^{it}}{\psi(e^{it}) - z}\right] dt \qquad z \in G. \tag{2.12}$$

Here $[\ ] = i +$ negative powers of e^{it} and hence its conjugate

$$\overline{\Big[\ \Big]} = -i + \sum_{k>0} d_k e^{ikt}.$$

Therefore by (2.11)

$$\frac{1}{2\pi i}\int_0^{2\pi} f(e^{it})\,\overline{\left[\frac{\psi'(e^{it})\,ie^{it}}{\psi(e^{it}) - z}\right]}\,dt = -if(0)\cdot(-i) = -f(0).$$

Subtracting this from (2.12) we get our *alternative representation* of the operator T:

$$F(z) = (Tf)(z) = \frac{1}{\pi}\int_0^{2\pi} f(e^{it})\,\operatorname{Im}\left[\frac{\psi'(e^{it})\,ie^{it}}{\psi(e^{it}) - z}\right] dt - f(0)$$

or

$$F(z) = (Tf)(z) = \frac{1}{\pi}\int_0^{2\pi} f(e^{it})\,\frac{d}{dt}\arg\{\psi(e^{it}) - z\}\,dt - f(0), \qquad z \in G; \tag{2.13}$$

see Korevaar [8], p. 288 with a somewhat different derivation.

If now Γ is of BSV, we have (2.2), and from (2.13) we obtain

$$|F(z)| \leqq \|f\|\cdot\frac{1}{\pi}\operatorname{Var}_t \arg\{\psi(e^{it}) - z\} + \|f\| \leqq \|f\|\cdot\left[\frac{M}{\pi} + 3\right].$$

THEOREM 3 (Andersson [2], Korevaar [8]). *If Γ is of BSV, the Faber operator is bounded.*

3. A NEW CONDITION FOR $\|T\| < \infty$

As we noted in Section 2.1, a smooth Jordan curve Γ need not be of BSV. However, we are now going to prove

THEOREM 4. *If the Jordan curve Γ is piecewise Dini-smooth, then $\Gamma \in BSV$.*

Combining this with Theorem 3 from above, this will prove Theorem 1. Notice that corners and cusps are permitted in Γ.

3.1. *Reduction of the Problem*

1. Let $\gamma: \zeta = \zeta(s)$ be a piecewise smooth Jordan arc, and let $z_0 \in \mathbb{C}$. We denote by

$$V(\gamma, z_0) = \operatorname{Var}_\zeta \arg(\zeta - z_0)$$

the total variation of $\arg(\zeta - z_0)$ as ζ traverses γ. This is an additive function of γ: If $\Gamma = \bigcup_j \gamma_j$ then

$$V(\Gamma, z_0) = \sum_j V(\gamma_j, z_0). \tag{3.1}$$

2. We now give a rough estimate. Again, let γ be piecewise smooth, with $|\zeta'(s)| \leqq m$ on γ and l as the length of γ. Assume that $\operatorname{dist}(\gamma, z_0) = r > 0$. If then

$$\theta(s) = \arg(\zeta(s) - z_0) = \operatorname{Im} \log(\zeta(s) - z_0)$$

we have

$$\theta'(s) = \operatorname{Im} \frac{\zeta'(s)}{\zeta(s) - z_0} \qquad \text{and hence} \quad |\theta'(s)| \leqq \frac{m}{r},$$

so that

$$V(\gamma, z_0) \leqq \frac{m}{r} l. \tag{3.2}$$

This means that an arc γ at a positive distance from z_0 gives only a bounded contribution to the secant variation with respect to z_0.

3. We now reduce our problem: It is sufficient to prove Theorem 4 for Dini-smooth Jordan curves Γ. So let Γ be piecewise Dini-smooth, and let $z_0 \in \Gamma$. We write

$$\Gamma = \bigcup_{j=-n}^{m} \gamma_j \qquad \text{and assume} \quad z_0 \in \gamma_0.$$

We may assume that any two adjacent arcs γ_j and γ_{j+1} form an angle $\neq \pi$. Because of (3.1) we have

$$V(\Gamma, z_0) = \sum_{j=-n}^{m} V(\gamma_j, z_0).$$

For $j > 1$ and $j < -1$ the arcs γ_j are at a distance $\operatorname{dist}(\gamma_j, z_0) \geqq r > 0$, with r depending on Γ only, so that (3.2) gives

$$V(\gamma_j, z_0) \leqq \frac{m_j}{r} l_j \qquad \text{for} \quad j > 1 \quad \text{and} \quad j < -1.$$

More critical are the cases $j = \pm 1$ and $j = 0$. To estimate $V(\gamma_1, z_0)$, we extend γ_1 by a Jordan arc γ_1' in such a way that

$\gamma_1 \cup \gamma_1'$ is a smooth Jordan curve Γ_1

γ_0 lies inside Γ_1 (except for the point $\gamma_0 \cap \gamma_1$).

Since γ_1 was Dini-smooth, γ_1' can obviously be chosen so that Γ_1 is Dini-smooth, too.

Assume now that we know that a Dini-smooth Jordan curve is of *BSV*. Then

$$V(\gamma_1, z_0) \leqq V(\Gamma_1, z_0) \leqq \sup\{ V(\Gamma_1, Q) : Q \in \Gamma_1\} + 2\pi$$

by an application of Lemma 1. Similarly we estimate $V(\gamma_{-1}, z_0)$. To estimate $V(\gamma_0, z_0)$, we extend γ_0 by γ_0' so that $\Gamma_0 = \gamma_0 \cup \gamma_0'$ is a Dini-smooth Jordan curve, and again

$$V(\gamma_0, z_0) \leqq V(\Gamma_0, z_0)$$

where now $z_0 \in \Gamma_0$.

4. Our reduced problem is therefore to show that a *Dini-smooth* Jordan curve Γ is of *BSV*. Because of (3.2) it suffices to show this for an arc around z_0, and even for a subarc γ of Γ with endpoint z_0 which we may choose to be the origin. This leads us to the following *final problem*:

Given a Dini-smooth arc γ: $z = z(s)$ with $0 \leqq s \leqq s_0$ where $z(0) = 0$ and $\arg z'(0) = 0$ (horizontal tangent at 0). We need to estimate the total variation of $\theta(s) = \arg z(s)$ in the interval $[0, s_0]$.

3.2. *Secant Variation of a Dini-smooth Arc*

We now come to the problem mentioned at the end of the last section. However, we represent the arc γ in a more suitable form.

THEOREM 5. *Let γ be a Dini-smooth Jordan arc*:

$$\gamma : z = z(x) = x + ih(x) \qquad 0 \leqq x \leqq x_0, \qquad \textit{with} \quad h(0) = 0, h'(0) = 0,$$

where h' is Dini-continuous, i.e. its modulus of continuity

$$\omega(t) = \omega(t, h') = \sup\{ |h'(x_1) - h'(x_2)| : |x_1 - x_2| \leqq t\}$$

satisfies

$$\int_{t=0}^{x_0} \frac{\omega(t)}{t}\, dt \leqq A < \infty. \tag{3.3}$$

Then the secant variation $V(\gamma, z_0)$ with respect to $z_0 = 0$ is

$$V(\gamma, 0) \leqq 2A. \tag{3.4}$$

Proof. If $\theta = \arg z(x)$, $0 < x \leqq x_0$, we have to estimate

$$V(\gamma, 0) = \int_\gamma |d\theta| = \int_0^{x_0} \left|\frac{d\theta}{dx}\right| dx \leqq \int_0^{x_0} \left|\left(\frac{h(x)}{x}\right)'\right| dx$$

since $\tan \theta = h(x)/x$ and therefore

$$\left|\frac{d\theta}{dx}\right| = \cos^2\theta \cdot \left|\left(\frac{h(x)}{x}\right)'\right| \leqq \left|\left(\frac{h(x)}{x}\right)'\right|.$$

Notice that

$$\left(\frac{h(x)}{x}\right)' = \frac{h'(x)}{x} - \frac{h(x)}{x^2},$$

in which $|h'(x)| \leqq \omega(x)$ and

$$|h(x)| = \left|\int_0^x h'(t)\, dt\right| \leqq \int_0^x \omega(t)\, dt \leqq x \cdot \omega(x).$$

Hence

$$\left|\left(\frac{h(x)}{x}\right)'\right| \leqq 2\,\frac{\omega(x)}{x},$$

and (3.4) follows. ∎

This completes the proof of Theorem 4.

4. A DOMAIN WITH UNBOUNDED FABER OPERATOR

4.1. *Preliminaries*

The Faber operator T maps the powers w^n onto the Faber polynomials $F_n(z)$, for $n = 1, 2, \ldots$. If T is a bounded operator from the Banach space $A(\overline{\mathbb{D}})$ to $A(\overline{G})$, we therefore have

$$\|F_n\|_{A(\overline{G})} \leqq \|T\| \cdot \|w^n\|_{A(\overline{\mathbb{D}})} = \|T\|,$$

that is the Faber polynomials must be uniformly bounded on $\overline{G}$. If this is so, we see from (2.8) that the sequence $\{nb_n\}_{n=1}^{\infty}$ is bounded, where b_n are the coefficients of the exterior mapping ψ. In order to produce a domain G with boundary Γ for which the Faber operator is *unbounded*, it suffices to construct Γ such that $\{nb_n\}$ is unbounded.

A Jordan curve Γ with this property was first produced by Clunie [5]. His construction actually gives a *quasiconformal* Jordan curve Γ for which the Faber operator is not bounded. Our main tool is Becker's univalence criterion; see below.

4.2. *The Exterior Mapping Function* ψ

Following Clunie, we define ψ so that $\log \psi'$ is represented by a gap power series:

$$\log \psi'(w) = \sum_{k=3}^{\infty} c_k w^{-q^k - 2}, \qquad |w| > 1 \tag{4.1}$$

with $q = 10$ and coefficients c_k with $|c_k| \leqq M = 1.01$. Clearly $\psi'(w) = \exp(\sum_{k=3}^{\infty} \ldots)$ and so

$$\psi(w) = w + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots, \qquad |w| > 1 \tag{4.2}$$

is holomorphic in $\{w: |w|>1\}$. To see that ψ is univalent, we apply Becker's criterion (Becker [3], p. 322; Pommerenke [10], p. 173): If

$$(|w|^2-1)\cdot\left|w\,\frac{\psi''(w)}{\psi'(w)}\right|\leqq\varrho<1\qquad\text{for}\quad |w|>1 \tag{4.3}$$

then ψ maps $\{w: |w|>1\}$ univalently onto the exterior of a quasiconformal curve Γ.

We show that (4.3) is satisfied with $\varrho=0.99$. First,

$$\left|w\,\frac{\psi''(w)}{\psi'(w)}\right|\leqq\sum_{k=3}^{\infty}|c_k|\,(q^k+2)\,y^{q^k+2}\leqq m\sum_{k=3}^{\infty}(q^k+2)\,y^{q^k+2}$$

where we have put $y=|w|^{-1}\in(0,1)$. Further,

$$|w|^2-1=y^{-2}-1\leqq 2(1-y)\,y^{-2}$$

and therefore the left hand side of (4.3) is

$$\leqq 2M(1-y)\sum_{k=3}^{\infty}q^k y^{q^k}+4M\sum_{k=3}^{\infty}(1-y)\,y^{q^k}. \tag{4.4}$$

In the *second* term $4M=4.04$, while each term in the series is $<q^{-k}$ so that

$$\sum_{k=3}^{\infty}(1-y)\,y^{q^k}<10^{-3}+10^{-4}+\cdots;$$

in other words, the second term in (4.4) is <0.0045.

In the *first* term of (4.4), the factor of M is less than

$$2\log\frac{1}{y}\cdot\sum_{k=1}^{\infty}q^k y^k=2q^{-j+x}\cdot\sum_{k=1}^{\infty}q^k\exp(-q^{-j+x+k})\leqq B_q<0.9699$$

where we have put $y=\exp(-q^{-j+x})$ and used the estimate in Pommerenke [11], p. 190. Altogether, the left hand side of (4.3) is less than

$$1.01B_{10}+0.0045<1.01\cdot 0.9699+0.0045<0.99.$$

Now we choose the coefficients in (4.1) to be constant: $c_k=c=1.01$ and put $l_k=q^k+2$ so that

$$\psi'(w)=\prod_{k=3}^{\infty}\exp(cw^{-l_k})$$
$$=\left(1+\frac{c}{w^{l_3}}+\cdots\right)\cdot\left(1+\frac{c}{w^{l_4}}+\cdots\right)\cdot\cdots\cdot\left(1+\frac{c}{w^{l_k}}+\cdots\right)\cdot\cdots.$$

Multiplying out, the coefficient of w^{-n}, for $n = l_3 + l_4 + \cdots + l_k$, is $\geqq c^{k-2}$. These coefficients in the expansion of ψ' are therefore unbounded, and we have proved:

If $c_k = c = 1.01$ in (4.1), the exterior mapping ψ in (4.2) will have coefficients b_n with nb_n unbounded, and ψ will map $\{w: |w| > 1\}$ onto a domain with a quasiconformal boundary Γ. By what we have said in Section 4.1, the finite domain G bounded by Γ will have an unbounded Faber operator. Theorem 2 is proved.

Remark. For more on schlicht functions with large coefficients, see Carleson and Jones [4].

4.3. *Refinement*

We saw that the Faber polynomials F_n associated with the curve Γ from above are not uniformly bounded on Γ. More is true: *For each fixed $z_0 \in \mathbb{C}$, the Faber polynomials are unbounded at z_0.* This was observed by Suetin ([13], p. 224) in connection with Clunie's example mentioned earlier.

To see this, we note that the sequence $\{nb_n\}$ is not only unbounded but closer inspection shows that

$$n\,|b_n| \geqq n^{\gamma} \qquad \text{for some} \quad \gamma > 0 \quad \text{and infinitely many } n. \tag{4.5}$$

Using (4.5), we can even show: *For each $z_0 \in \mathbb{C}$, and for each $p < \gamma$ with γ from* (4.5), *the sequence $\{F_n(z_0) \cdot n^{-p}\}$ is unbounded.*

To see this, we use the recursion formula

$$(n+1)\,b_n = (z_0 - b_0)\,F_n(z_0) - F_{n+1}(z_0) - \sum_{k=1}^{n-1} b_{n-k} F_k(z_0) \qquad (n = 1, 2, ...);$$

see Pommerenke [10], p. 57. If $|F_n(z_0)| \leqq Mn^p$ for all n and some M, then

$$n\,|b_n| \leqq An^p + B \sum_{k=1}^{n-1} |b_{n-k}|\, k^p = An^p + B \sum_{k=1}^{n-1} |b_k| \sqrt{k} \cdot \frac{(n-k)^p}{\sqrt{k}},$$

in which the sum is bounded by

$$\left[\sum_{k=1}^{n-1} |b_k|^2\, k\right]^{1/2} \cdot \left[\sum_{k=1}^{n-1} \frac{(n-k)^{2p}}{k}\right]^{1/2} \leqq 1 \cdot n^p [1 + \log n]^{1/2}$$

by the area theorem. Hence $nb_n = \mathcal{O}(n^p \sqrt{\log n})(n \to \infty)$. If $p < \gamma$, this contradicts (4.5) so that $\{F_n(z_0) \cdot n^{-p}\}$ cannot be bounded.

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